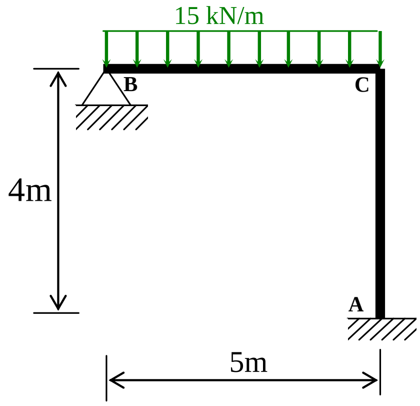
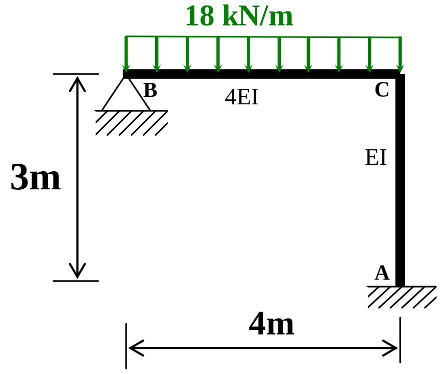
Exercícios - Utilize o **Método dos deslocamentos** para calcular as reações de apoio e trace os diagramas de esforços normal, cortante e momento fletor dos quadros hiperestáticos:

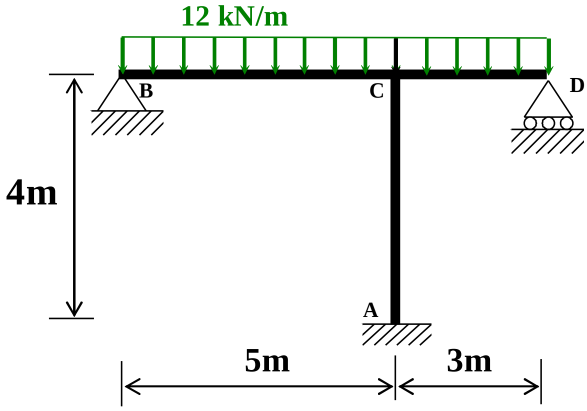
1)



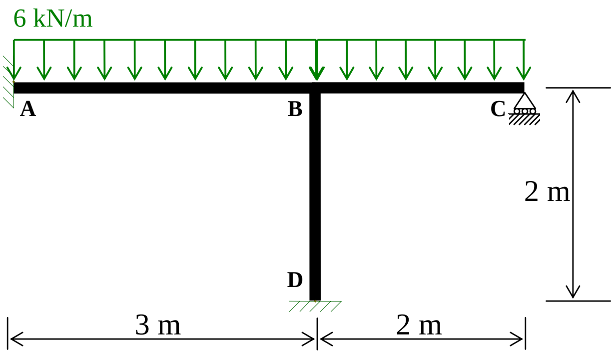
2)



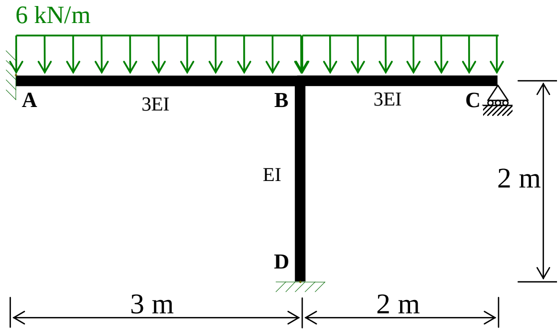
3)



4)



5)



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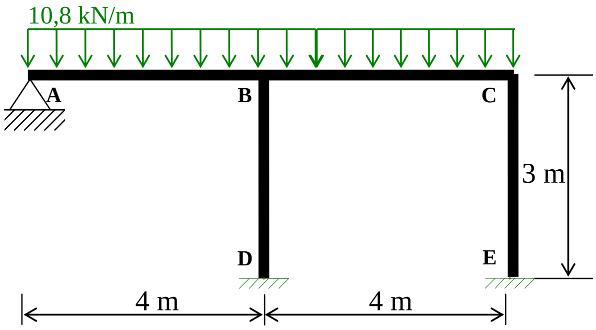

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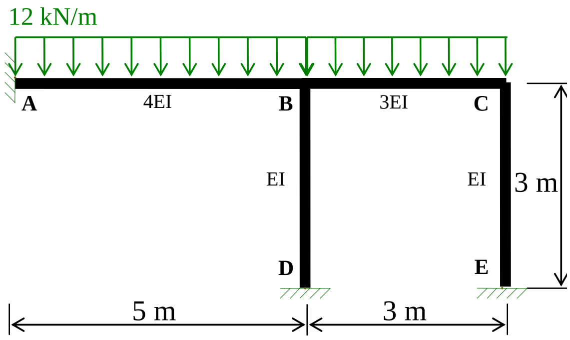

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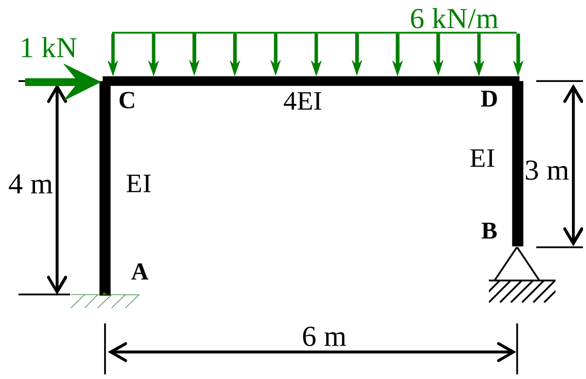
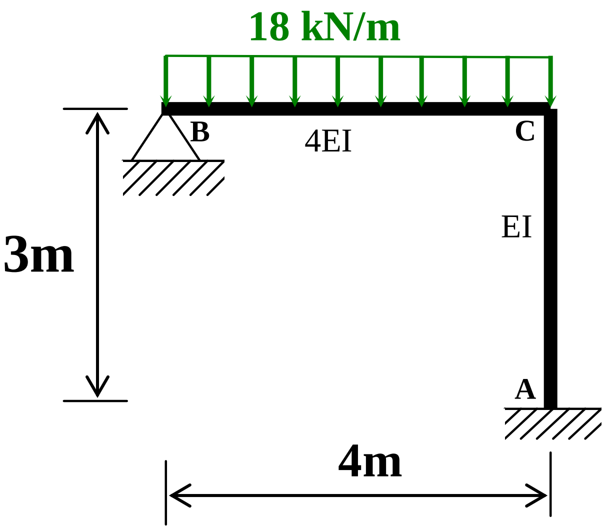


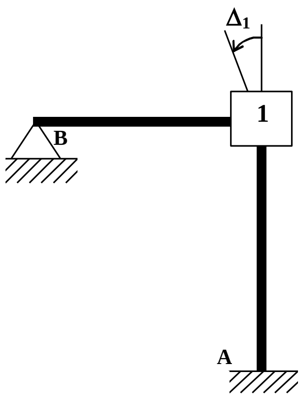
Obs.: Confirme as reações de apoio e os esforços com o software:
Ftool → http://www.tecgraf.puc-rio.br/ftool/

2)

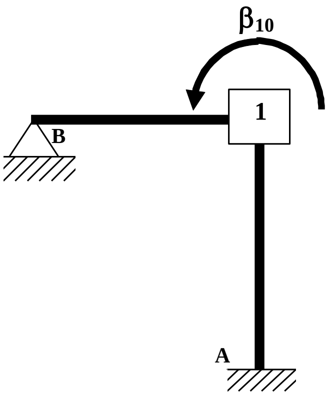


Solução:

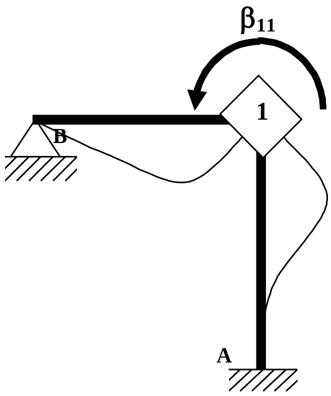
1- Sistema Principal



2- Efeitos no sistema principal



Carregamento Externo



Rotação Δ_1

barra BC:

$$M_{B1} = -\frac{qL_1^2}{8} = -\frac{18x4^2}{8} = -36$$

Temos então:

$$\beta_{10} = M_{B1} = -36$$

barra BC:

$$k_{C1} = \frac{3EI}{L_1} = \frac{3x4EI}{4} = 3EI$$

barra AC:

$$k_{C2} = \frac{4EI}{L_2} = \frac{4EI}{3} = 1{,}333EI$$

Temos então:

$$\beta_{11} = k_{C1} + k_{C2} \Rightarrow$$

$$\beta_{11} = 3EI + 1{,}333EI = 4{,}333EI$$

3- Cálculo da incógnita Δ_1

Sabemos que:

$$\beta_{10} + \beta_{11}\Delta_1 = 0 \Rightarrow \Delta_1 = -\frac{\beta_{10}}{\beta_{11}} \Rightarrow \Delta_1 = -\frac{-36}{4{,}333EI}$$

$$\therefore \boxed{\Delta_1 = \frac{8{,}30769}{EI}}$$

4- Reações de Apoio

$$V_B = V_B^o + \Delta_1 V_B^1 \Rightarrow$$

$$V_B = \frac{3qL_1}{8} + \Delta_1 \frac{3EI}{L_1^2} \Rightarrow$$

$$V_B = \frac{3 \times 18 \times 4}{8} + \left(\frac{8{,}30769}{EI}\right)\frac{3 \times 4EI}{4^2} \Rightarrow$$

$$\boxed{V_B = 33{,}23 \text{ kN}}$$

$$H_A = H_A^o + \Delta_1 H_A^1 \Rightarrow$$

$$H_A = 0 + \Delta_1 \frac{6EI}{L_2^2} \Rightarrow$$

$$H_A = 0 + \left(\frac{8{,}30769}{EI}\right)\frac{6EI}{3^2} \Rightarrow$$

$$\boxed{H_A = 5{,}54 \text{ kN}}$$

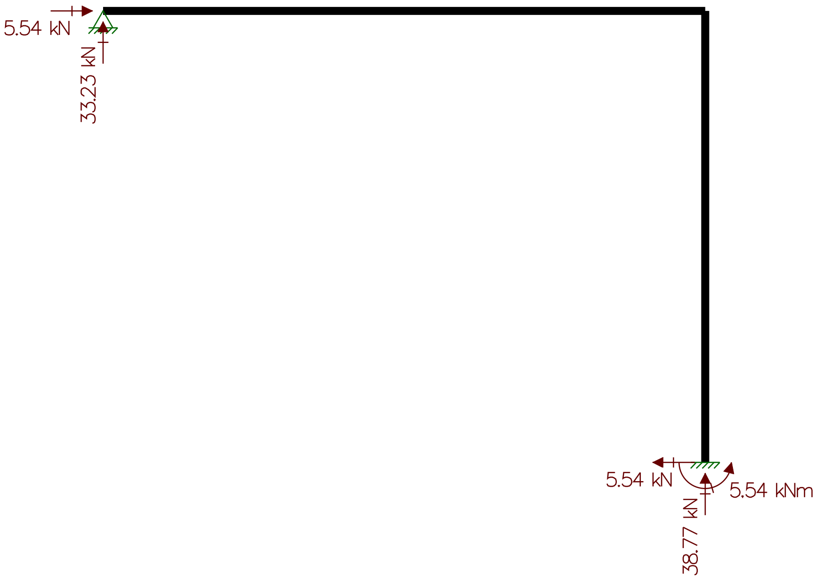
$$M_A = M_A^o + M_A^1 \Delta_1 \Rightarrow$$

$$M_A = 0 + \frac{2EJ}{L_2}\Delta_1 \Rightarrow$$

$$M_A = 0 + \frac{2EJ}{3}\left(\frac{8{,}30769}{EJ}\right) \Rightarrow$$

$$\boxed{M_A = 5{,}54 \text{ kNm}}$$

Representação gráfica das reações de apoio



As demais reações de apoio podem ser calculadas por equilíbrio estático.

$$\sum F_y = 0 \Rightarrow$$

$$33{,}23 + V_A - 18x4 = 0 \Rightarrow$$

$$\boxed{V_A = 38{,}77 \text{ kN}}$$

$$\sum F_x = 0 \Rightarrow$$

$$5{,}54 - H_B = 0 \Rightarrow$$

$$\boxed{H_B = 5{,}54 \text{ kN}}$$

5- Diagramas de esforços

Barra BC

Equações com origem em B (x=0).

$$\boxed{0 \le x \le 4\,m}$$

$$N(x) = -H_B$$

$$V(x) = V_B - 18x$$

$$M(x) = V_B x - 9x^2$$

Barra AC

Equações com origem em A (x=0).

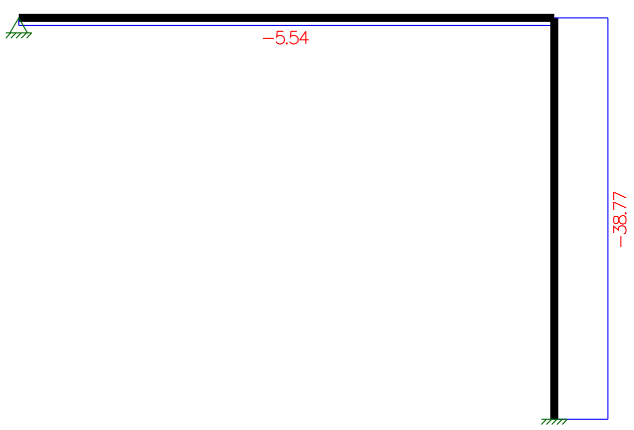
$$\boxed{0 \le x \le 3\,m}$$

$$N(x) = -V_A$$

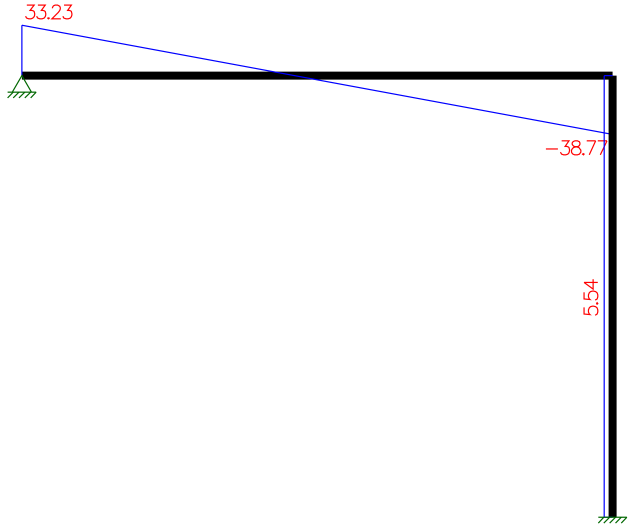
$$V(x) = H_A$$

$$M(x) = M_A - H_A x$$

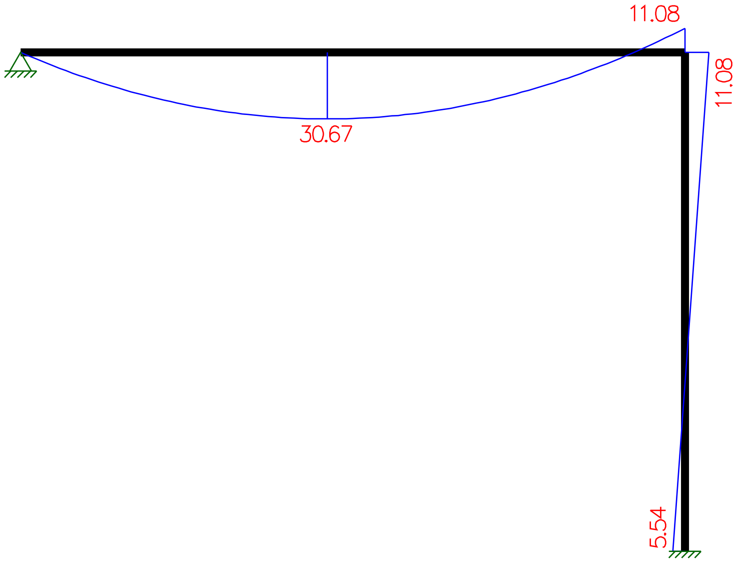
a) Esforço Normal



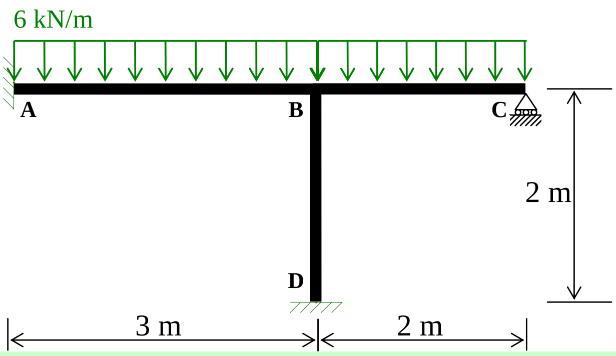
b) Cortante



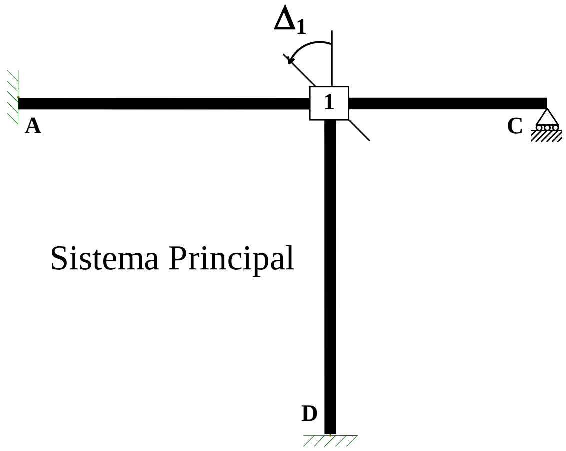
c) Momento Fletor

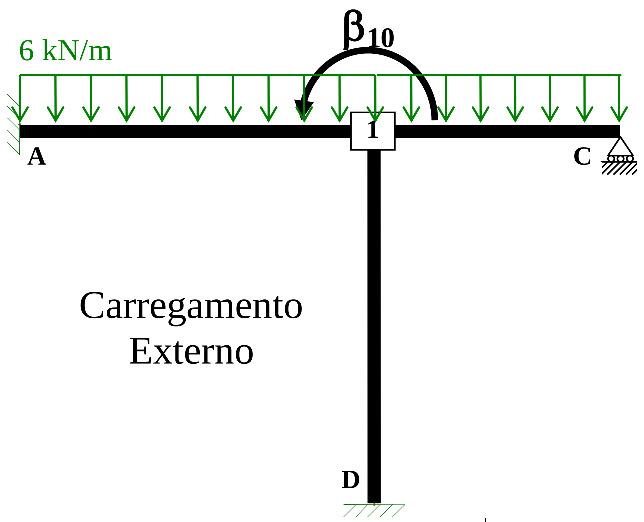


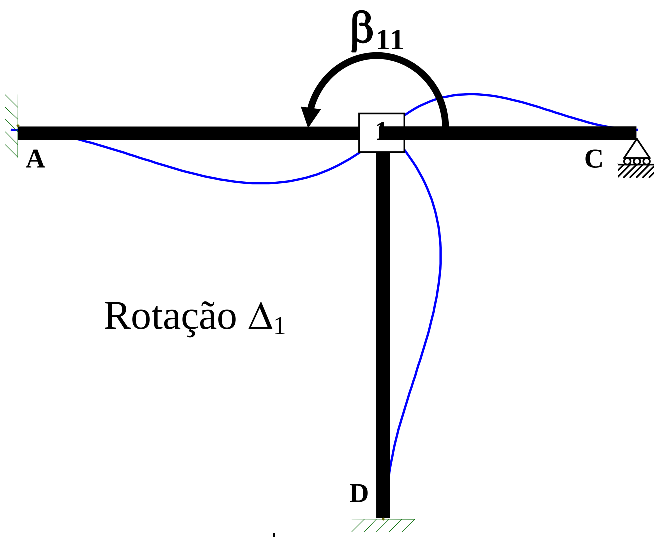
4)



Solução:







Carregamento Externo:

$$M_{B1} = -\frac{qL_{AB}^2}{12} = -\frac{6\times 3^2}{12} = -\frac{9}{2}$$

$$M_{B2} = \frac{qL_{BC}^2}{8} = \frac{6\times 2^2}{8} = 3$$

Temos então:

$$\beta_{10} = M_{B1} + M_{B2} = -\frac{9}{2} + 3 = -\frac{3}{2}$$

Rotação Δ_1

$$k_{B1} = \frac{4EI}{L_{AB}} = \frac{4\times EI}{3} = \frac{4}{3}EI$$

$$k_{B2} = \frac{3EI}{L_{BC}} = \frac{3\times EI}{2} = \frac{3}{2}EI$$

$$k_{B3} = \frac{4EI}{L_{BD}} = \frac{4\times EI}{2} = 2EI$$

Temos então:

$$\beta_{11} = \frac{4}{3}EI + \frac{3}{2}EI + 2EI = \frac{29}{6}EI$$

Cálculo da incógnita Δ_1

Sabemos que:

$$\beta_{10} + \beta_{11}\Delta_1 = 0 \Rightarrow$$

$$\Delta_1 = -\frac{\beta_{10}}{\beta_{11}} \Rightarrow$$

$$\Delta_1 = -\frac{-\frac{3}{2}}{\frac{29}{6}EI}$$

$$\therefore \boxed{\Delta_1 = \frac{9}{29EI}}$$

4- Reações de Apoio

$$V_A = V_A^o + \Delta_1 V_A^1$$

$$\Rightarrow V_A = \frac{qL_{AB}}{2} + \Delta_1 \frac{6\times EI}{L_{AB}^2}$$

$$\Rightarrow V_A = \frac{6\times 3}{2} + \left(\frac{9}{29EI}\right)\frac{6\times EI}{3^2} = \frac{267}{29}$$

$$\therefore V_A = 9{,}207\ \text{kN}$$

$$M_A = M_A^o + \Delta_1 M_A^1$$

$$\Rightarrow M_A = \frac{qL_{AB}^2}{12} + \Delta_1 \frac{2\times EI}{L_{AB}}$$

$$\Rightarrow M_A = \frac{6\times 3^2}{12} + \left(\frac{9}{29EI}\right)\frac{2\times EI}{3} = \frac{273}{58}$$

$$\therefore M_A = 4{,}707\ \text{kNm}$$

$$V_D = V_D^o + \Delta_1 V_D^1$$

$$\Rightarrow V_D = \left(\frac{qL_{AB}}{2} + \frac{5qL_{BC}}{8}\right) + \Delta_1\left[-\frac{6EI}{L_{AB}^2} + \frac{3EI}{L_{BC}^2}\right]$$

$$\Rightarrow V_D = \frac{6\times 3}{2} + \frac{5\times 6\times 2}{8} + \left(\frac{9}{29EI}\right)\left[-\frac{6\times EI}{3^2} + \frac{3\times EI}{2^2}\right]$$

$$\therefore V_D = 16{,}526\ \text{kN}$$

$$M_D = M_D^o + \Delta_1 M_D^1$$

$$\Rightarrow M_D = 0 + \Delta_1 \frac{2EI}{L_{BD}}$$

$$\Rightarrow M_D = 0 + \left(\frac{9}{29EI}\right)\frac{2EI}{2}$$

$$\therefore M_D = 0{,}310\ \text{kNm}$$

$$V_C = V_C^o + \Delta_1 V_C^1$$

$$\Rightarrow V_C = \frac{3qL_{BC}}{8} + \Delta_1\left[-\frac{3EI}{L_{BC}^2}\right]$$

$$\Rightarrow V_C = \frac{3\times 6\times 2}{8} + \left(\frac{9}{29EI}\right)\left[-\frac{3EI}{2^2}\right]$$

$$\therefore V_C = 4{,}267\ \text{kN}$$

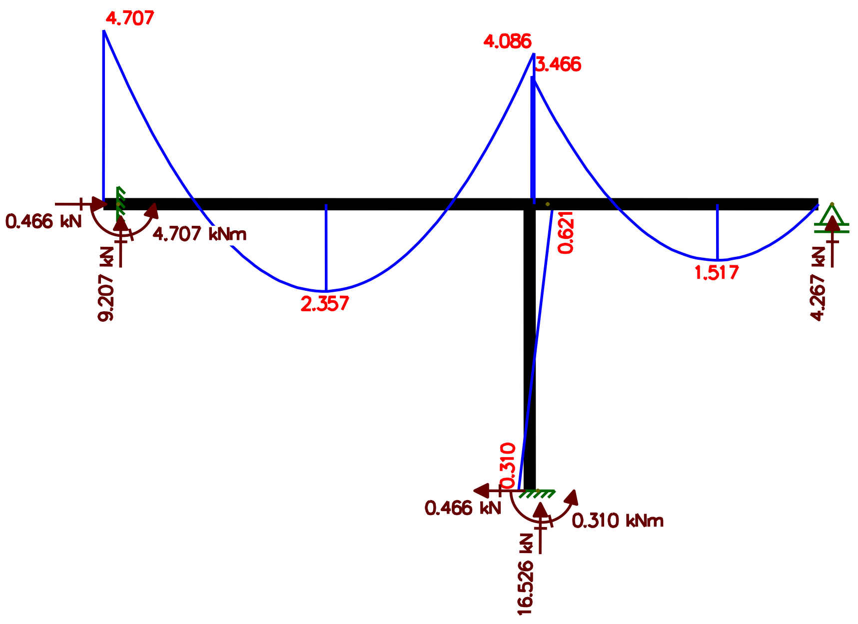
$$H_D = H_D^o + \Delta_1 H_D^1$$

$$\Rightarrow H_D = 0 + \Delta_1\left[-\frac{6\times EI}{L_{BD}^2}\right]$$

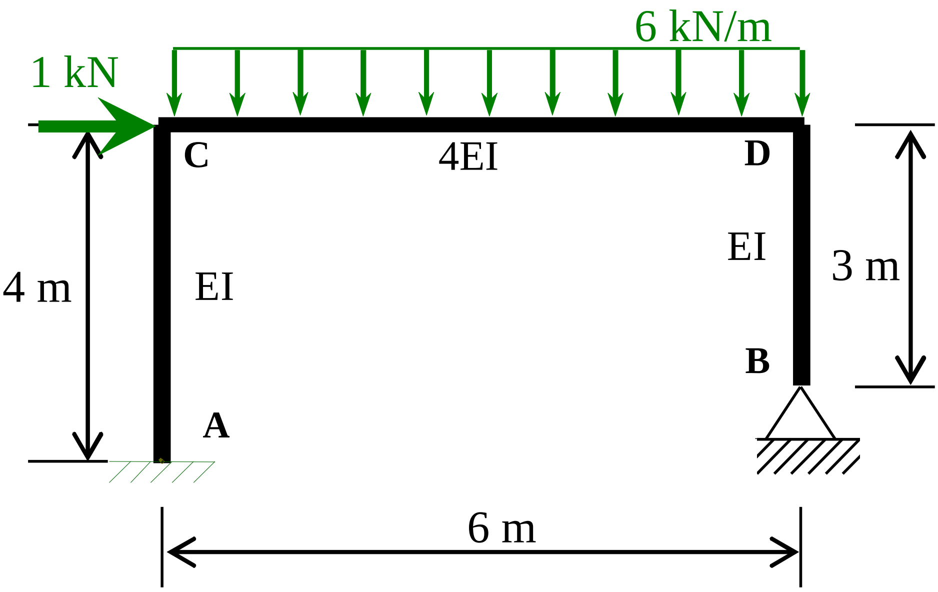
$$\Rightarrow H_D = 0 + \left(\frac{9}{29EI}\right)\frac{-6EI}{2^2}$$

$$\therefore H_D = -0{,}466\ \text{kN}$$

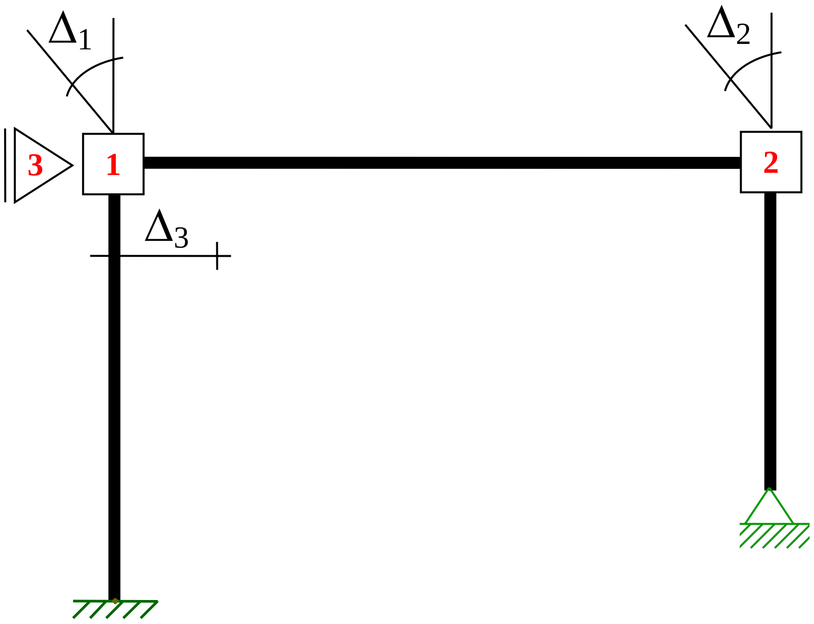
$$\therefore H_A = 0{,}466\ \text{kN}$$



8)

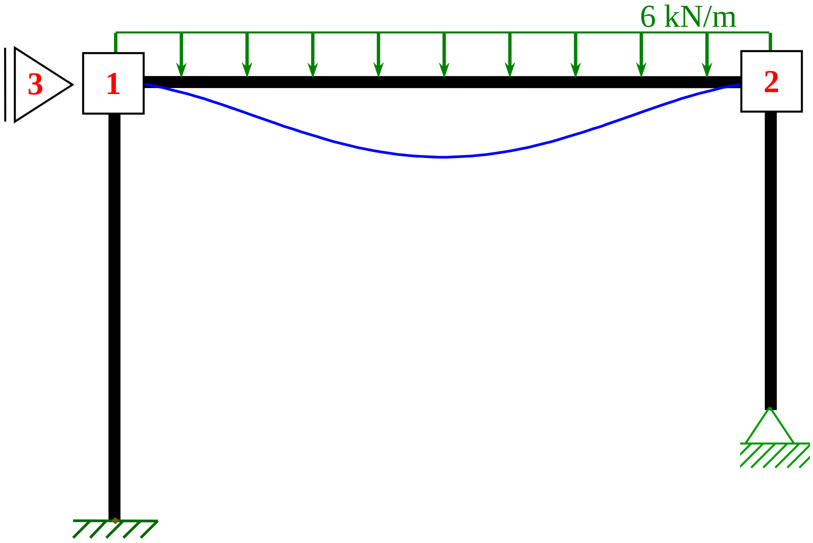


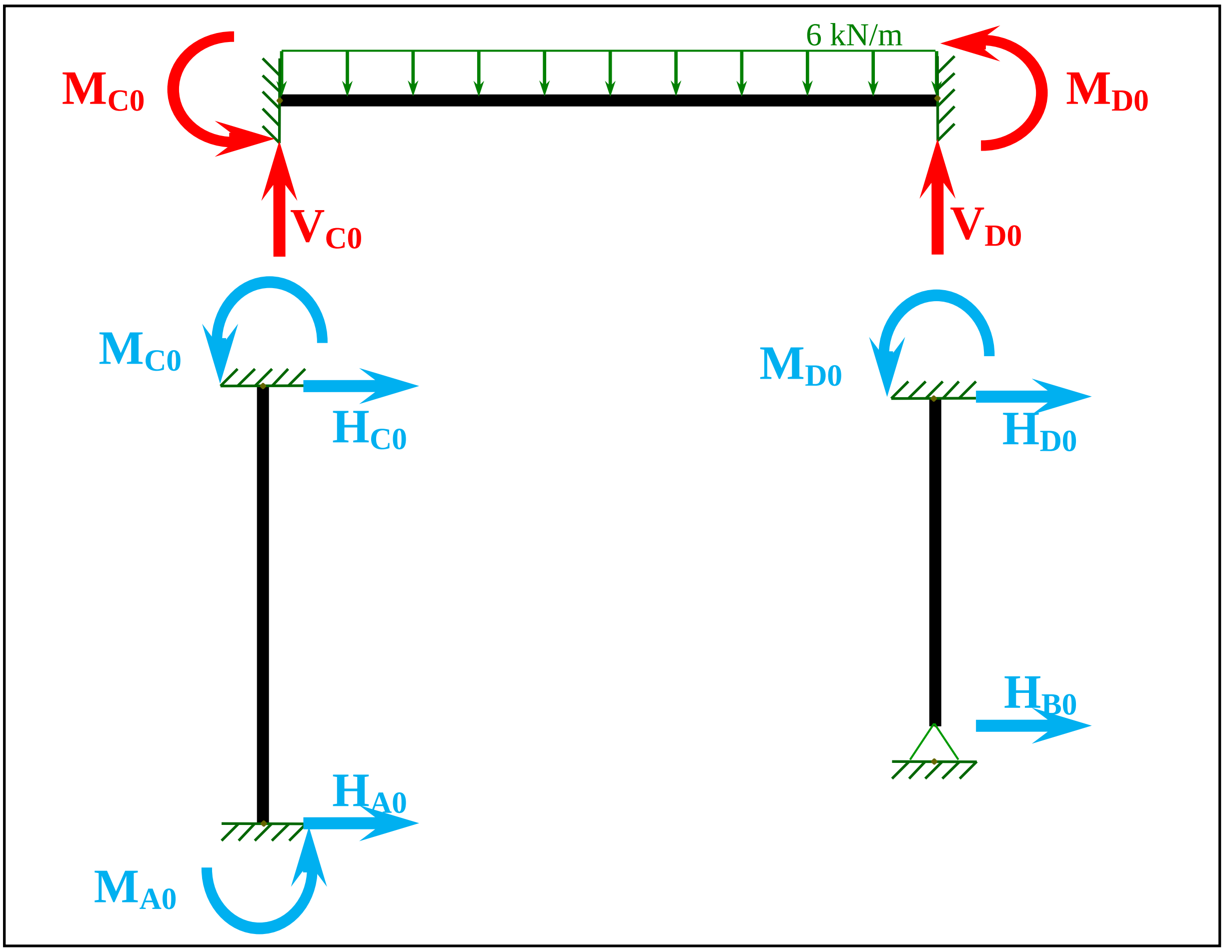
Sistema Principal



Δ_1 = Rotação do nó C (deslocabilidade interna)
Δ_2 = Rotação do nó D (deslocabilidade interna)
Δ_3 = Translação da direção CD (deslocabilidade externa)

Direção 0 – Carregamento original





Para a barra CD temos os seguintes valores:

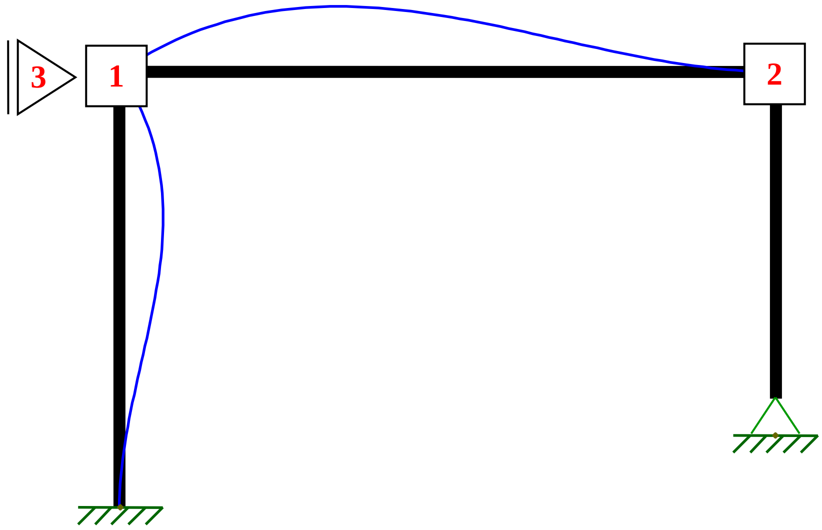
$$M_{C0} = \frac{q\ell^2}{12} = \frac{6 \times 6^2}{12} = 18$$

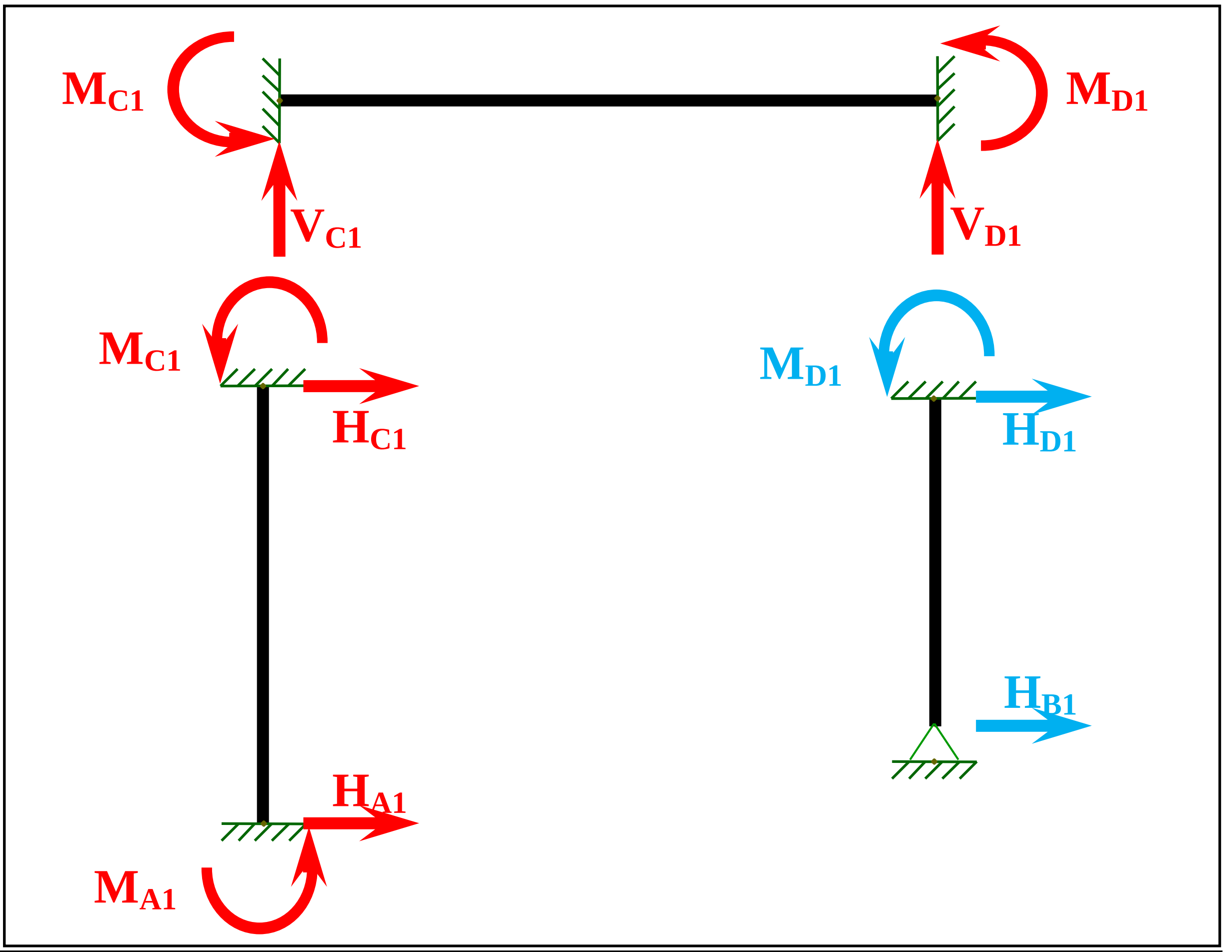
$$M_{D0} = -\frac{q\ell^2}{12} = -\frac{6 \times 6^2}{12} = -18$$

$$V_{C0} = V_{D0} = \frac{q\ell}{2} = \frac{6 \times 6}{2} = 18$$

Os demais valores (na cor **azul**) são todos iguais a zero.

Direção 1 – Rotação unitária na direção de Δ_1

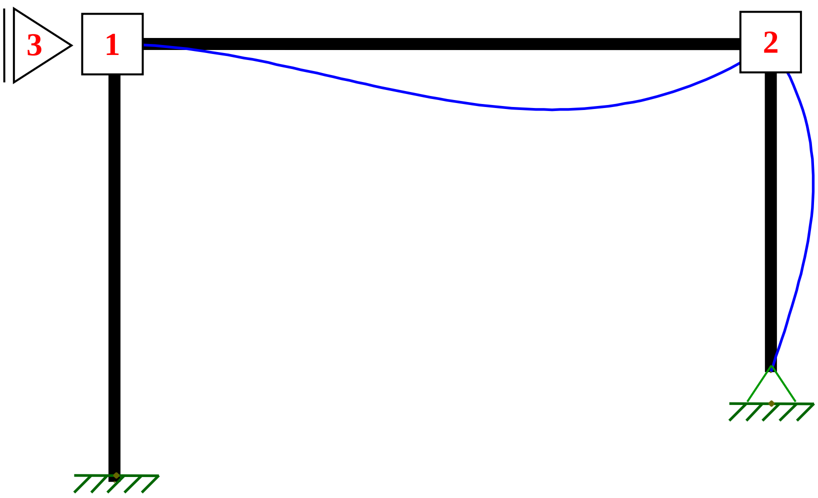


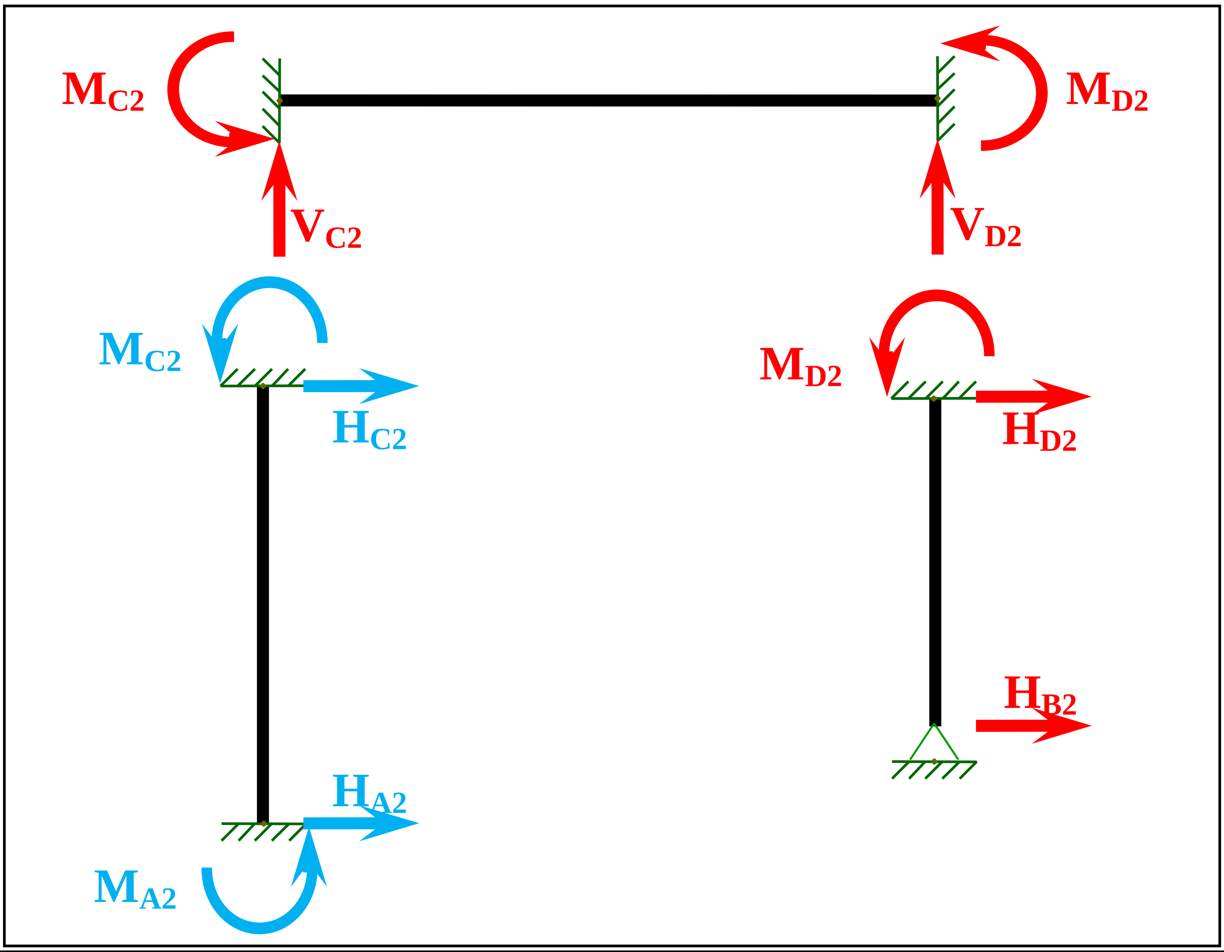


Para a barra CD temos os seguintes valores:	Para a barra AC temos os seguintes valores:
$M_{C1} = \frac{4(4EI)}{\ell} = \frac{4(4EI)}{6} = \frac{16EI}{6} = \frac{8EI}{3}$	$M_{C1} = \frac{4EI}{\ell} = \frac{4EI}{4} = EI$
$M_{D1} = \frac{2(4EI)}{\ell} = \frac{2(4EI)}{6} = \frac{8EI}{6} = \frac{4EI}{3}$	$M_{A1} = \frac{2EI}{\ell} = \frac{2EI}{4} = \frac{EI}{2}$
$V_{C1} = -V_{D1} = \frac{6(4EI)}{\ell^2} = \frac{24EI}{6^2} = \frac{24EI}{36} = \frac{2EI}{3}$	$H_{C1} = -H_{A1} = \frac{6EI}{\ell^2} = \frac{6EI}{4^2} = \frac{6EI}{16} = \frac{3EI}{8}$

Os demais valores (na cor **azul**) são todos iguais a zero.

Direção 2 – Rotação unitária na direção de Δ_2

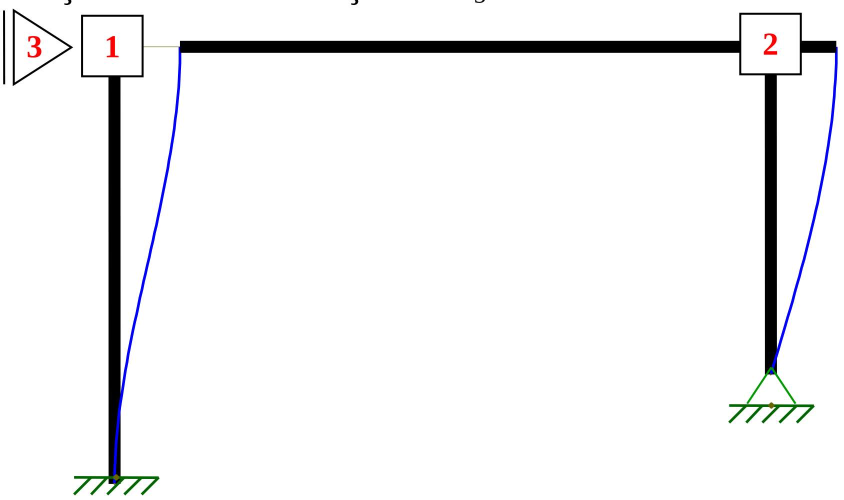


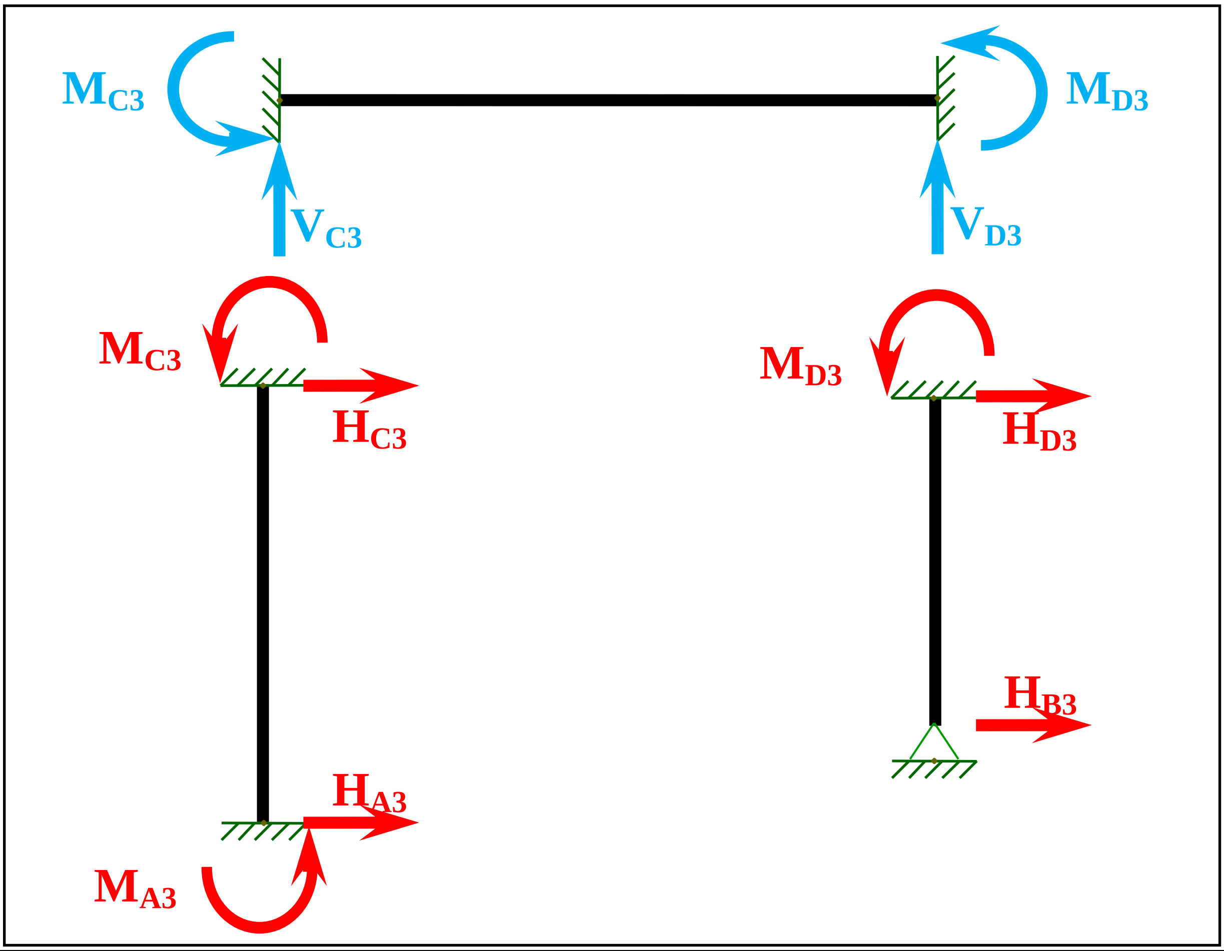


Para a barra CD temos os seguintes valores:	Para a barra BD temos os seguintes valores:
$M_{C2}=\frac{2(4EI)}{\ell}=\frac{2(4EI)}{6}=\frac{8EI}{6}=\frac{4EI}{3}$	$M_{D2}=\frac{3EI}{\ell}=\frac{3EI}{3}=EI$
$M_{D2}=\frac{4(4EI)}{\ell}=\frac{4(4EI)}{6}=\frac{16EI}{6}=\frac{8EI}{3}$	$H_{D2}=-H_{B2}=\frac{3EI}{\ell^2}=\frac{3EI}{3^2}=\frac{3EI}{9}=\frac{EI}{3}$
$V_{C2}=-V_{D2}=\frac{6(4EI)}{\ell^2}=\frac{24EI}{6^2}=\frac{24EI}{36}=\frac{2EI}{3}$	

Os demais valores (na cor **azul**) são todos iguais a zero.

Direção 3 – Translação unitária na direção de Δ_3





Para a barra AC temos os seguintes valores:	Para a barra BD temos os seguintes valores:
$M_{C3} = M_{A3} = \frac{6EI}{\ell^2} = \frac{6EI}{4^2} = \frac{6EI}{16} = \frac{3EI}{8}$	$M_{D3} = \frac{3EI}{\ell^2} = \frac{3EI}{3^2} = \frac{3EI}{9} = \frac{EI}{3}$
$H_{C3} = -H_{A3} = \frac{12EI}{\ell^3} = \frac{12EI}{4^3} = \frac{12EI}{64} = \frac{3EI}{16}$	$H_{D3} = -H_{B3} = \frac{3EI}{\ell^3} = \frac{3EI}{3^3} = \frac{3EI}{27} = \frac{EI}{9}$

Os demais valores (na cor **azul**) são todos iguais a zero.

Equações de compatibilidade

$$\sum M_1 = 0 \Rightarrow \quad \beta_{10} + \beta_{11}\Delta_1 + \beta_{12}\Delta_2 + \beta_{13}\Delta_3 = 0$$

$$\sum M_2 = 0 \Rightarrow \quad \beta_{20} + \beta_{21}\Delta_1 + \beta_{22}\Delta_2 + \beta_{23}\Delta_3 = 0$$

$$\sum H_3 = 1 \Rightarrow \quad \beta_{30} + \beta_{31}\Delta_1 + \beta_{32}\Delta_2 + \beta_{33}\Delta_3 = 1$$

Onde:

$$\beta_{10} = M_{C0} = 18$$

$$\beta_{20} = M_{D0} = -18$$

$$\beta_{30} = 0$$

$$\beta_{11} = M_{C1} + M_{C1} = \frac{8EI}{3} + EI = \frac{11EI}{3}$$

$$\beta_{21} = M_{D1} = \frac{4EI}{3}$$

$$\beta_{31} = H_{C1} = \frac{3EI}{8}$$

$$\beta_{12} = \beta_{21}$$

$$\beta_{22} = M_{D2} + M_{D2} = \frac{8EI}{3} + EI = \frac{11EI}{3}$$

$$\beta_{32} = H_{D2} = \frac{EI}{3}$$

$$\beta_{13} = \beta_{31}$$

$$\beta_{23} = \beta_{32}$$

$$\beta_{33} = H_{C3} + H_{D3} = \frac{3EI}{16} + \frac{EI}{9} = \frac{43EI}{144}$$

Assim:

$$\begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{vmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = -\begin{Bmatrix} \beta_{10} \\ \beta_{20} \\ \beta_{30} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Rightarrow \qquad EI \begin{vmatrix} \frac{11}{3} & \frac{4}{3} & \frac{3}{8} \\ \frac{4}{3} & \frac{11}{3} & \frac{1}{3} \\ \frac{3}{8} & \frac{1}{3} & \frac{43}{144} \end{vmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{Bmatrix} -18 \\ 18 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{Bmatrix} -8{,}1391722 \\ 7{,}3845231 \\ 5{,}3269346 \end{Bmatrix} \frac{1}{EI}$$

Reações de Apoio:

$$V_A = V_{A0} + \Delta_1 V_{A1} + \Delta_2 V_{A2} + \Delta_3 V_{A3} \Rightarrow \qquad V_A = 17{,}4969 \text{ kN}$$

$$H_A = H_{A0} + \Delta_1 H_{A1} + \Delta_2 H_{A2} + \Delta_3 H_{A3} \Rightarrow \qquad H_A = 2{,}0539 \text{ kN}$$

$$M_A = M_{A0} + \Delta_1 M_{A1} + \Delta_2 M_{A2} + \Delta_3 M_{A3} \Rightarrow \quad M_A = -2{,}0720 \text{ kN}$$

$$V_B = V_{B0} + \Delta_1 V_{B1} + \Delta_2 V_{B2} + \Delta_3 V_{B3} \Rightarrow \qquad V_B = 18{,}5031 \text{ kN}$$

$$H_B = H_{B0} + \Delta_1 H_{B1} + \Delta_2 H_{B2} + \Delta_3 H_{B3} \Rightarrow \qquad H_B = -3{,}0539 \text{ kN}$$